Math 72: 8.3 & 8.4 Solving Quadratic Equations

Objectives:

1) Solve equations by the square root property.
2) Solve equations using completing the square.
3) Solve quadratic equations using the quadratic formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4) Use the discriminant $D = b^2 - 4ac$ to determine the number and type of solutions.
5) Summarize the methods for solving quadratic equations.
6) Solve application problems involving quadratic equations.
    a. Solve an equation for a specified variable.
    b. Find the x-intercepts of a quadratic function $f(x) = ax^2 + bx + c$
    c. Solve uniform motion (D=RT) problems.
    d. Use position functions.
    e. Geometry problems

Examples

Simplify:

1) $\dfrac{9}{10} - \sqrt{\dfrac{-361}{100}}$ 2) $\dfrac{12 - 8\sqrt{7}}{16}$ 3) $\dfrac{10 - 20\sqrt{-288}}{16}$

4) Solve by factoring, then by the square root property: $x^2 - 81 = 0$
5) Solve by the square root property.
    a. $2x^2 - 24 = 0$
    b. $(2x-5)^2 = -16$
6) $3p^2 - 6p = 4$
    a. Solve by CTS & square root property
    b. Identify the number and type of solutions using the previous work.
7) Solve for the specified variable:
    a. $E = mv^2$ for v
    b. $a^2 = c^2 - b^2$ for b
8) Solve by the QF. Identify the number and type of solutions.
    a. $3x^2 - 9x + 8 = 0$
    b. $6x^2 - 17x - 14 = 0$
    c. $25x^2 - 20x + 4 = 0$
9) Without solving the equation, calculate only the discriminant and use it to determine the number and type of solutions.
    a. $2x^2 - 4x = 3$
    b. $m^2 - \dfrac{m}{2} + \dfrac{1}{16} = 0$
    c. $\dfrac{x}{3} = -x^2 - 1$

10) An investment of $2000 grows to $2420 when compounded annually for two years. What is the interest rate?

11) Find all intercepts of

a. $f(x) = 2(x-1)^2 + 3$

b. $f(x) = \frac{1}{2}x^2 - 2x - 1$

12) Together, an experienced word processor and an apprentice word processor can create a document in 6 hours. Alone, the experienced one can do the job 2 hours faster than the apprentice can do it working alone. Find the time for each to do the job alone.

13) Beach and Fargo are 400 miles apart. A salesperson travels from Beach to Fargo on Monday, and back to Beach on Tuesday. On Tuesday, she drives 10 mph faster. The total time spent driving was $14\frac{2}{3}$ hours. Find her speed in each direction. Round to the nearest mile per hour, if necessary.

14) Suppose that an open box is to be made from a square sheet of cardboard by cutting 2-inch-by-2-inch squares out of each corner and folding the sides up. The finished box has volume 128 cubic inches. Find he original dimensions of the sheet of cardboard.

15) Students cut diagonally across the lawn instead of walking two sidewalks which form a right angle. The path on the lawn is 20 feet, while they travel x feet and x+2 feet on the two sidewalks. How many feet do they save by cutting across the lawn?

16) A water tank can be filled by the large and small inlet pipes in 3 hours. The large inlet pipe can fill the tank in 2 hours less time than the small inlet pipe. Find the time, to the nearest tenth of an hour, that each pipe takes to fill the tank alone.

17) Bill and his son Billy can clean the house together in 4 hours. Billy takes an hour longer by himself than Bill takes by himself. How long, to the nearest hour, does it take each one to clean the house, if working alone?

18) At the 2007 Grand Prix of Long Beach auto race, Simon Pagenaud posted the fastest lap speed, but Sebastian Bourdais won the race. One lap through the streets of Long Beach is 10,391 feet (1.968 miles) long. Pagenaud's fastest lap speed was 0.55 foot per second faster than Bourdais's fastest lap speed. Traveling at these fastest speeds, Bourdais would have taken 0.25 second longer than Pagenaud to complete a lap.

a. Find Sebastian Bourdais's fastest lap speed during the race. Round to two decimal places.

b. Find Simon Pagenaud's fastest lap speed during the race. Round to two decimal places.

c. Convert each speed to miles per hour. Round to one decimal place.

19) A family drives 500 miles to the Grand Canyon. The return trip was made at an average speed 10 mph faster. The total traveling time was $18\frac{1}{3}$ hours. Find the speed to the Grand Canyon and the return speed.

Sign of the discriminant	**Perfect Square**	**Number of solutions**	**Type of solutions**
Positive	Perfect square*	2	Real, rational
	Not a perfect square	2	Real, irrational
Zero*	Doesn't matter	1	Real, rational
Negative	Doesn't matter	2	Complex

*If the discriminant is a positive perfect square or zero, the equation can be factored.

Methods for Solving Quadratic Equations

<table>
<tr><th>Method</th><th>Advantages</th><th>Disadvantages</th><th>Things to remember</th></tr>
<tr><td>Factoring
using zero-product property
$(x+e)(x-d)=0$
$x=-e,d$</td><td>• Some equations factor easily.
$x^2-6x-7=0$
$x^2-6x=0$
• Factoring can be used to solve higher-order equations.</td><td>• Some equations are difficult to factor.
$72x^2-41x-117=0$
• Other equations cannot be factored at all.
$x^2-6x+7=0$</td><td>• The right-hand-side of the equation must be zero.</td></tr>
<tr><td>Square root property
$x^2=n$
$x=\pm\sqrt{n}$</td><td>• This is the fastest way to solve equations in the forms:
$x^2=3 \quad 4(x-1)^2-8=0$
$3x^2=12 \quad (4x+1)^2=9$</td><td></td><td>• Isolate the square first.
• Must include ± when taking the square root of both sides.
• RHS is not usually zero.</td></tr>
<tr><td>Completing the square
followed by the square root property.
$ax^2+bx+c=0$
$\left(x+\frac{b}{2}\right)^2=-\frac{c}{a}+\left(\frac{b}{2}\right)^2$</td><td>• Can be used to solve any quadratic equation.
• Does not usually involve simplifying complicated square roots.
• GC's MATH >frac makes the fraction calculations easy.
• Is used to create the quadratic formula.</td><td>• The process has several steps which must be done in order.
• Fractions result when $a\neq 1$ or b is not even.</td><td>• The leading coefficient must be 1, so divide entire equation by any leading coefficient.
• RHS is not zero.
• Must add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.
• The factor is always $\left(x+\frac{b}{2}\right)^2$</td></tr>
<tr><td>Quadratic formula
$ax^2+bx+c=0$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$</td><td>• Can be used to solve any quadratic equation for all solutions.</td><td>• Always requires simplifying a complicated square root.
• Must memorize the formula.</td><td>• Equation must be fully simplified (no parentheses) and in standard form.
• RHS must equal zero.
• Entire formula is divided by 2a, not just the radical.</td></tr>
<tr><td>Graphing</td><td>• Use to check work and gain intuition.</td><td>• Approximate.
• Must calculate twice if there are two answers.
• Won't help if answers are complex.</td><td>• Intersection method – two graphs, one for LHS and one for RHS.
• x-Intercept method – one graph of LHS-RHS or RHS-LHS.</td></tr>
</table>

Simplify.

No 1 $\dfrac{9}{10} - \sqrt{\dfrac{-361}{100}}$

$= \dfrac{9}{10} - \dfrac{\sqrt{-361}}{\sqrt{100}}$ quotient property of square roots

$= \dfrac{9}{10} - \dfrac{\sqrt{-1} \cdot \sqrt{361}}{\sqrt{100}}$ product property of square roots

$= \dfrac{9}{10} - \dfrac{i \;\; 19}{10}$

$19^2 = 361$
$10^2 = 100$
$i = \sqrt{-1}$ by definition

$= \boxed{\dfrac{9}{10} - \dfrac{19}{10} i}$

$\underset{a}{\uparrow} \quad + \quad \underset{b \cdot i}{\uparrow}$

standard form for complex numbers is a+bi form, separate numbers for real part a and imaginary part b.

CAUTION: MML/MXL will accept $\dfrac{9-19i}{10}$ even though this is nonstandard notation.

Do NOT do this on a PQ or Exam!

No 2 $\dfrac{12 - 8\sqrt{7}}{16}$

$= \dfrac{12}{16} - \dfrac{8\sqrt{7}}{16}$ divide each term by denominator

$= \boxed{\dfrac{3}{4} - \dfrac{\sqrt{7}}{16}}$

Option 2: Factor numerator

$\dfrac{4(3 - 2\sqrt{7})}{16}$

$= \boxed{\dfrac{3 - 2\sqrt{7}}{4}}$

Note: Neither contains i, so not a complex number. Either answer is acceptable.

YES (3) $\dfrac{10 - 20\sqrt{-288}}{16}$

Several tasks must be completed, but can be done in any order

1. simplify $\sqrt{-1}$
2. simplify $\sqrt{288}$
3. Because $\sqrt{-1} = i$, must be $a+bi$ form. Divide both terms by 16.

$= \dfrac{10}{16} - \dfrac{20}{16} i\sqrt{288}$

simplify $\sqrt{-1} = i$
divide both terms by 16

$= \dfrac{5}{8} - \dfrac{5}{4} i \cdot \sqrt{144 \cdot 2}$

288 → 144, 2

prime factors of 288
OR
divide by the biggest perfect square.

4	49
9	64
16	81
25	100
36	121
	144

$= \dfrac{5}{8} - \dfrac{5}{4} i\sqrt{144} \cdot \sqrt{2}$

product property of radicals

$= \dfrac{5}{8} - \dfrac{5}{4} i (12)\sqrt{2}$

reduce $\dfrac{12}{4} = 3$

$= \dfrac{5}{8} - 5 \cdot 3 i\sqrt{2}$

$= \boxed{\dfrac{5}{8} - 15 i\sqrt{2}}$ or $\boxed{\dfrac{5}{8} - 15\sqrt{2}\, i}$

Caution: If using this format, i must be clearly outside $\sqrt{\ }$

$\sqrt{i}$ is not what you mean, and it does mean something else.

$\sqrt{i} = \sqrt{\sqrt{-1}} = \sqrt[4]{-1}$

NO 4 $x^2-81=0$

Solve by factoring the difference of squares

$(x-9)(x+9)=0$

$\boxed{x=9 \quad x=-9}$

There are two solutions. Any correct method must find both.

Solve by square root property
- step 1: Isolate the perfect square.
- step 2: Square root both sides, simplify $\sqrt{\ }$.
- step 3: Include $\pm$ to capture both solutions unless rational, then simplify.

$x^2-81=0$

$x^2=81$ — isolate square = get x^2 alone

$\sqrt{x^2}=\sqrt{81}$

$|x|=9$ — technically, $\sqrt{x^2}=|x|$, which is where the $\pm$ comes from!

$\boxed{x=\pm 9}$

YES 5 a) $2x^2-24=0$

$2x^2=24$

$x^2=12$ — isolate x^2

$x=\pm\sqrt{12}$

$x=\pm\sqrt{4\cdot 3}$ — square root property

$\boxed{x=\pm 2\sqrt{3}}$ — simplify radical

b) $(2x-5)^2=-16$ — The square $(2x-5)^2$ is already isolated.

$2x-5=\pm\sqrt{-16}$

$2x-5=\pm\sqrt{-1}\sqrt{16}$ — square root property

$2x-5=\pm 4i$ — simplify radical

$2x=5\pm 4i$ — isolate x

$x=\frac{5}{2}\pm\frac{4}{2}i$ — divide each to get a+bi form

$\boxed{x=\frac{5}{2}\pm 2i}$ — simplify

YES (6) $3p^2 - 6p = 4$

CANNOT EVER USE CTS WHEN LEADING COEFFICIENT $\neq 1$

Method 1: Divide all terms by leading coefficient, even this creates fractions throughout.

$$p^2 - 2p = \frac{4}{3}$$

To complete the square take middle coefficient, divide by 2, and square result.

$\# = \frac{-2}{2} = -1$ ← This value is factor value

$\#^2 = (-1)^2 = 1$ ← Add this to both sides

$p^2 - 2p + 1 = \frac{4}{3} + 1$ ← add $\#^2$ to both sides

$(p-1)^2 = \frac{7}{3}$ ← use factor # to write perfect sq. on LHS

$p - 1 = \pm \frac{\sqrt{7} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$ ← rationalize

$\boxed{p = 1 \pm \frac{\sqrt{21}}{3}}$ or $\boxed{\frac{3 \pm \sqrt{21}}{3}}$

There are two solutions $1 + \frac{\sqrt{21}}{3}$ and $1 - \frac{\sqrt{21}}{3}$

(or $\frac{3+\sqrt{21}}{3}$ and $\frac{3-\sqrt{21}}{3}$).

$\sqrt{21}$ is an irrational number, but real. (no i)

So there are $\boxed{\text{2 solutions irrational, real}}$

Factor Out Leading Coefficient

Process for Solving by Completing the Square

step 1: Arrange equation with all variables on LHS and constant on RHS.

(6) $3p^2 - 6p = 4$

step 2: Factor out the leading coefficient (or divide both sides by it)

$3p^2 - 6p = 4$

$3(p^2 - 2p \quad) = 4$

$\frac{3p^2}{3} - \frac{6p}{3} = \frac{4}{3}$

$p^2 - 2p = \frac{4}{3}$

step 3: Find the coefficient of the degree 1 term and divide it by 2.
This number will be used in step 6 to write the perfect square.

$\# = \frac{-2}{2} = -1$

step 4: Square the result from step 3.

$\#^2 = (-1)^2 = 1$

step 5: Add the result from step 4 inside parentheses on LHS of equation.
Mentally distribute and add result to RHS of equation.

$\underbrace{3(p^2 - 2p + 1)}_{3 \cdot 1 = 3} = 4 + 3$

$3(p^2 - 2p + 1) = 7$

step 6: Use the result in step 3 to factor the quantity in parentheses to form a perfect square.

$3(p-1)^2 = 7$

step 7: Solve using the square root property

$(p-1)^2 = \frac{7}{3}$

$p - 1 = \pm\sqrt{\frac{7}{3}}$

$p - 1 = \pm\frac{\sqrt{7} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$

$p - 1 = \frac{\pm\sqrt{21}}{3}$

$\boxed{p = 1 \pm \frac{\sqrt{21}}{3}}$

Check $p = \frac{3+\sqrt{21}}{3}$ by evaluating by hand. Easier $1+\frac{\sqrt{21}}{3}$.

$3p^2 - 6p = 4$

substitute for p.

$$3\left(1+\frac{\sqrt{21}}{3}\right)^2 - 6\left(1+\frac{\sqrt{21}}{3}\right) = 4$$

$$3\left(1+\frac{\sqrt{21}}{3}\right)\left(1+\frac{\sqrt{21}}{3}\right) - 6 - 2\sqrt{21} = 4$$

$$3\left(1 + 2\cdot 1\cdot\frac{\sqrt{21}}{3} + \frac{21}{9}\right) - 6 - 2\sqrt{21} = 4$$

$$3 + \frac{6\sqrt{21}}{3} + \frac{21}{3} - 6 - 2\sqrt{21} = 4$$

$$3 + 2\sqrt{21} + 7 - 6 - 2\sqrt{21} = 4$$

$$4 = 4$$ ☺

Check $p = \frac{3-\sqrt{21}}{3} = 1-\frac{\sqrt{21}}{3}$ by hand

$$3\left(1-\frac{\sqrt{21}}{3}\right)^2 - 6\left(1-\frac{\sqrt{21}}{3}\right) = 4$$

$$3\left(1-\frac{\sqrt{21}}{3}\right)\left(1-\frac{\sqrt{21}}{3}\right) - 6 + 2\sqrt{21} = 4$$

$$3\left(1 - 2\cdot\frac{\sqrt{21}}{3} + \frac{21}{9}\right) - 6 + 2\sqrt{21} = 4$$

$$3 - 2\sqrt{21} + 7 - 6 + 2\sqrt{21} = 4$$

$$4 = 4$$ ☺

Both $p = 1+\frac{\sqrt{21}}{3}$ and $p = 1-\frac{\sqrt{21}}{3}$ are solutions of $3p^2 - 6p = 4$.

Check $p = \dfrac{3+\sqrt{21}}{3}$ by evaluating function by GC:

In GC $(3+\sqrt{}(21))/3$ $\boxed{\text{STO}\blacktriangleright}$ $\boxed{\text{ALPHA}}$ $\boxed{8}$ $\boxed{\text{ENTER}}$

"p" location

This stores all the decimal places in memory location P.

2.527525232

In GC $\boxed{Y=}$ $Y_1 = 3x^2 - 6x - 4$

In GC $\boxed{\text{VARS}}$ $\boxed{\triangleright}$

To $\boxed{\text{Y-VARS}}$ MENU

$\boxed{1}$ Function

$\boxed{1}$ Y_1

In screen

Y_1

Type $\boxed{(}$ $\underbrace{\boxed{\text{ALPHA}}\ \boxed{8}}_{\text{"p"}}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

This evaluates Y_1 when X = value stored in memory location P.

$Y_1(P)$	
	0

This value of P <u>IS</u> a solution of Y_1.

$\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$ $\Rightarrow$ gives back what we typed 2 entries ago.

Use $\blacktriangleleft$ to move on top of $+$, then type $\boxed{-}$, and press $\boxed{\text{ENTER}}$

$(3-\sqrt{}(21))/3 \to P$ stores $-.5275252317$ in location P

$\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$ $\Rightarrow$ gives back what we typed 2 entries ago

$Y_1(P)$

$\boxed{\text{ENTER}}$

← Now it's using the new value stored in memory location P.

$(3-\sqrt{}(21))/3\to P$	
	$-.5275252317$
$Y_1(P)$	
	0

$Y_1(P)$ means

$$3\left(\frac{3-\sqrt{21}}{3}\right)^2 - 6\left(\frac{3-\sqrt{21}}{3}\right) - 4 = 0$$

confirming that $p = \dfrac{3-\sqrt{21}}{3}$ is also a solution.

<u>Note</u>: You can also type in the number instead of using memory location values.

YES 7 a) Solve $E = mv^2$ for v

isolate v using the reverse order of operations

have multiply → undo by divide by m

exponent 2 → undo by square root property

$$\frac{E}{m} = \frac{mv^2}{m}$$

$$\frac{E}{m} = v^2$$

$$\boxed{\pm\sqrt{\frac{E}{m}} = v}$$

b) $a^2 = c^2 - b^2$ for b

isolate b using reverse order of operations

have add c^2 → undo by subtract c^2

have mult by -1 ⇒ undo by divide by -1

have exp b^2 ⇒ undo by square root property

$$\begin{aligned} a^2 &= c^2 - b^2 \\ -c^2 & \quad\; -c^2 \end{aligned}$$

$$\frac{a^2}{-1} - \frac{c^2}{-1} = \frac{-b^2}{-1}$$

$$-a^2 + c^2 = b^2$$

$$\boxed{\pm\sqrt{-a^2+c^2} = b}$$

or $\boxed{b = \pm\sqrt{c^2 - a^2}}$

Process for solving by the Quadratic Formula (QF)

Memorize the quadratic formula for $ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

step 1: Set equation $=0$ and write in standard form $ax^2+bx+c=0$.

step 2: Identify the coefficients a, b, c.

step 3: Substitute values of a, b, c into formula.

step 4: Simplify result.

Process for Checking Solutions using graph on GC

The graph shows real solutions only. If answer has i, the graph can only confirm that the solutions have imaginary parts, but not the actual numerical values.

step 1: Set equation equal to zero

step 2: Graph the expression using $y=$

step 3: Find the x-intercepts of graph using "zero" (or "root")

step 4: If necessary, calculate the approximate values of the solutions found by CTS or QF so they can be compared to solutions found in step 3.

YES ⑧ $3x^2 - 9x + 8 = 0$ (2 complex) QF

a) $a = 3$ $b = -9$ $c = 8$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(3)(8)}}{2(3)}$$

$$x = \frac{9 \pm \sqrt{81 - 96}}{6}$$

$$x = \frac{9 \pm \sqrt{-15}}{6}$$

$$x = \frac{9}{6} \pm \frac{i\sqrt{15}}{6}$$

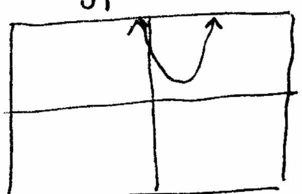
$$\boxed{x = \frac{3}{2} \pm i\frac{\sqrt{15}}{6}}$$

2 Complex solutions

a+bi form

Graph

$y_1 = 3x^2 - 9x + 8$



no x-ints

⇒ no real solutions

#2 We cannot check the numerical values of a complex number using a graph because it shows real numbers only

Check $x = \frac{3}{2} \pm i\frac{\sqrt{15}}{6}$ are solutions of $3x^2 - 9x + 8 = 0$

by evaluating the function.

$\boxed{Y=}$ $3x^2 - 9x + 8$

$3/2 +$ $\boxed{\text{2nd}}$ $\boxed{\cdot}$ $\sqrt{\ }(15)/6$ $\boxed{\text{STO>}}$ $\underbrace{\boxed{\text{ALPHA}}\ \boxed{\text{STO>}}}$

2nd decimal pt is $i = \sqrt{-1}$

$\boxed{\text{ALPHA}}\boxed{\text{STO>}}$ is memory location X

```
3/2+i√(15)/6→X
    1.5+.6454972244i
```

$\boxed{\text{VARS}}$ $\boxed{\triangleright}$ $\boxed{\text{Y VARS}}$ 1. Function
1. Y_1

$\boxed{(}$ $\underbrace{\boxed{\text{ALPHA}}\ \boxed{\text{STO>}}}_{\text{This is X}}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

This doesn't work for complex numbers ☹

ERROR: DATA TYPE.

$$= 3\left(\frac{3}{2} + i\frac{\sqrt{15}}{6}\right)^2 - 9\left(\frac{3}{2} + i\frac{\sqrt{15}}{6}\right) + 8$$

$$= 3\left(\frac{3}{2} + i\frac{\sqrt{15}}{6}\right)\left(\frac{3}{2} + i\frac{\sqrt{15}}{6}\right) - \frac{27}{2} - \frac{3}{2}i\sqrt{15} + 8$$

$$= 3\left(\frac{9}{4} + 2 \cdot \frac{3}{2} \cdot i\frac{\sqrt{15}}{6} - \frac{15}{36}\right) - \frac{11}{2} - \frac{3}{2}i\sqrt{15}$$

$$= \frac{27}{4} + \frac{3}{2}i\sqrt{15} - \frac{5}{4} - \frac{11}{2} - \frac{3}{2}i\sqrt{15}$$

$$= 0$$ ☺

You can do it by hand

$3/2 +$ $\boxed{\text{2nd}}$ $\boxed{\cdot}$ $\sqrt{\ }(15)/6$ $\boxed{\text{STO>}}$ $\underbrace{\boxed{\text{ALPHA}}\ \boxed{\text{STO}}}_{X}$

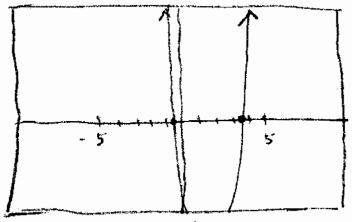
3 $\boxed{\text{ALPHA}}$ $\boxed{\text{STO>}}$ $\boxed{x^2}$ $- 9$ $\boxed{\text{ALPHA}}$ $\boxed{\text{STO>}}$ $+ 8$ $\boxed{\text{ENTER}}$

```
3/2+i√(15)/6→X
    1.5+.6454972244i
3X²-9X+8
                 0
```

or you can type it into GC yourself (no y= menu)

b) $6x^2 - 17x - 14 = 0$ (2 rational) Method 2: Quadratic Formula

$a = 6 \quad b = -17 \quad c = -14$



2 x ints at $x = 7/2, -2/3$

$$x = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(6)(-14)}}{2(6)}$$

$$x = \frac{17 \pm \sqrt{625}}{12}$$

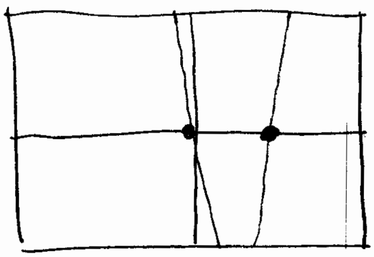
$$x = \frac{17 \pm 25}{12}$$

2 real rational solutions

$$x = \frac{17+25}{12}, \frac{17-25}{12} = \boxed{\frac{7}{2}, \frac{-2}{3}}$$

Graph

$y = 6x^2 - 17x - 14$



$\boxed{\text{2nd}}$ TRACE = CALC

2. zero

$x \approx -.6666667 \Rightarrow -.\overline{6} = \frac{-2}{3}$

$x \approx 3.5 \Rightarrow x = \frac{7}{2}$

c) $25x^2 - 20x + 4 = 0$ (1 rational)

$a = 25 \quad b = -20 \quad c = 4$

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(25)(4)}}{2(25)}$$

$$x = \frac{20 \pm \sqrt{0}}{50}$$

$\boxed{x = \frac{2}{5}}$ one real rational solution

Identifying Number and Type of Solutions

- If there's $i=\sqrt{-1}$ in the answer, solutions are complex numbers.
  Complex solutions do not appear on the graph.
  * If solutions are complex, there are always two $\pm$

- If there's no $i=\sqrt{-1}$ in the answer, solutions are real numbers.
  Real solutions appear as x-intercepts on graph.
  * If $\sqrt{0}=0$ appeared in QF work, there is only one solution
  * If $\sqrt{\text{positive}}$ appeared in QF work, there are two solutions. $\pm$

- Real solutions may be rational or irrational
  Irrational solutions occur when $\sqrt{\text{positive}}$ remains in solutions.
  * If solutions are real and irrational, there are two solutions. $\pm$

  Rational solutions occur when $\sqrt{\text{perfect square}}$ simplifies.
  * If perfect square is 0, there is only one solution.
  * If perfect square is not 0, there are two solutions. $\pm$
  If solutions are rational, equations can also be solved by factoring.

In $3p^2-6p-4=0$, there are two real and irrational solutions, $1+\frac{\sqrt{21}}{3}$ and $1-\frac{\sqrt{21}}{3}$

Without solving the equation, calculate only the discriminant and use it to determine the number and type of solutions. D = discriminant = b^2-4ac No square root!
Must be $ax^2+bx+c=0$

$3x^2+16x+5=0$

$D=b^2-4ac$

$16^2-4(3)(5) = \boxed{196} = 14^2$

$D>0 \Rightarrow$ 2 real solutions
D = perfect square $\Rightarrow$ $\boxed{\text{2 real, rational solutions}}$

9) a) $2x^2-4x=3 \Rightarrow 2x^2-4x-3=0$

$(-4)^2-4(2)(-3)$

$\boxed{D=40}$

$D>0 \Rightarrow$ 2 real solutions
D $\neq$ perfect square $\Rightarrow$ $\boxed{\text{2 real, irrational solutions}}$

b) $m^2-\frac{m}{2}+\frac{1}{16}=0$ clear fractions

$16m^2-8m+1=0$

$(-8)^2-4(16)(1) = \boxed{0=D}$

$D=0 \Rightarrow$ $\boxed{\text{1 real rational solution}}$

c) $\frac{x}{3}=-x^2-1$ clear fractions

$x=-3x^2-1$ set $=0$

$3x^2+x+1=0$

$1^2-4(3)(1) = \boxed{-11=D}$

$D<0 \Rightarrow$ $\boxed{\text{2 complex solutions}}$

Number and type of Solutions (flow chart)

calculate discriminant $D = b^2 - 4ac$ (no square root)

$D\ (+)$ $\Downarrow$ Real

- Real $\swarrow$ Rational: $\sqrt{\text{perfect sq}} = D$
  - $\sqrt{0}$ $\Downarrow$ one real rational solution
  - other $\Downarrow$ two real rational Solutions
- Real $\searrow$ Irrational: $\sqrt{\text{something that won't reduce}} = D$
  - $\Downarrow$ two real irrational solutions

$D\ (-)$ $\Downarrow$ Complex $\Downarrow$ 2 complex Solutions

* Memorize Discriminant for $ax^2+bx+c=0$ is $D = b^2-4ac$ (no $\sqrt{\ }$)

e.g.

$D = -23$
means
2 complex solutions

e.g $D = 23$
means
2 real irrational solutions.

e.g $D = 25$
2 real rational solutions

e.g. $D = 0$
1 real (repeated) rational solution

NO

(10) An investment of \$2000 grows to \$2420 when compounded annually for two years. What is the interest rate?

Compounded annually $\Rightarrow$ compound interest formula

$$A = P\left(1+\frac{r}{n}\right)^{(nt)}$$

A = amount in account
P = principal, initial amount
r = annual interest rate, decimal form
t = time in years
n = # times compounded per year

$A = 2420$
$P = 2000$
$r = ?$
$t = 2$
$n = 1$

$$2420 = 2000\left(1+\frac{r}{1}\right)^{(1\cdot 2)}$$ ✓ plug in

$$2420 = 2000(1+r)^2$$ ✓ simplify $\frac{r}{1} = r$ and $1\cdot 2 = 2$

$$2000(1+r)^2 = 2420$$ ✓ Swap LHS and RHS to clarify. What's left is a quadratic equation with square already completed!

$$(1+r)^2 = \frac{2420}{2000}$$

Divide both sides by 2000 to isolate the square.

$$(1+r)^2 = \frac{121}{100}$$

$\boxed{\text{MATH}}$ > frac is your friend!

$$1+r = \pm\sqrt{\frac{121}{100}}$$ ← square root property

$$1+r = \pm\frac{11}{10}$$

$$r = -1 \pm \frac{11}{10}$$

Convert decimals to percents.
$.1 = 10\%$
$-2.1 = -210\%$

$$r = -1+\frac{11}{10},\quad -1-\frac{11}{10}$$

negative interest rate does not make sense. This is an extraneous solution.

$$r = .1,\ -2.1$$

Disregard -210% soln.

$$\boxed{r = 10\%}$$

YES (11) "Find all intercepts" means "Find all x-intercepts" AND "Find all y-intercepts."

Recall: x-intercept is a point $(x,0)$ where the graph crosses the x-axis
y-intercept is a point $(0,y)$ where the graph crosses the y-axis.

a) $f(x) = 2(x-1)^2+3$

x intercept: set $0=y$ and solve for x.

$0 = 2(x-1)^2+3$ — since we have a perfect square let's use the square root property!

$-3 = 2(x-1)^2$

$\frac{-3}{2} = (x-1)^2$ — isolate the square

$\pm\sqrt{\frac{-3}{2}} = x-1$ — isolate x

$1 \pm \sqrt{\frac{-3}{2}} = x$

$x = 1 \pm \frac{i\sqrt{3}}{\sqrt{2}}$ imaginary part!

This graph has $\boxed{\underline{\text{no}}\text{ x-intercepts.}}$ Imaginary parts can't be graphed on the x-y plane.

y-intercept: set $x=0$ and solve for y

$f(0) = 2(0-1)^2+3$

$= 2(1)+3$

$= 5$ $\boxed{\text{y-intercept } (0,5)}$

Note: vertex $(1,3)$ is NOT the x-intercept OR the y-intercept!

b) $f(x) = \frac{1}{2}x^2-2x-1$

x-intercepts: set $y=0$

$0 = \frac{1}{2}x^2-2x-1$ — now it's an equation, so we can multiply by 2 to clear fractions

$2\cdot 0 = 2\cdot\frac{1}{2}x^2 - 2\cdot 2x - 2\cdot 1$

$0 = x^2-4x-2$

$x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-2)}}{2(1)}$

$x = \frac{4\pm\sqrt{16+8}}{2}$

$x = \frac{4\pm\sqrt{24}}{2}$ $24 \to 4 \cdot 6$

$$x = \frac{4 \pm \sqrt{4 \cdot 6}}{2}$$

$$x = \frac{4 \pm \sqrt{4} \cdot \sqrt{6}}{2}$$

$$x = \frac{4 \pm 2\sqrt{6}}{2}$$

$$x = \frac{4}{2} \pm \frac{2\sqrt{6}}{2}$$

$$x = 2 \pm \sqrt{6}$$

$$\boxed{\text{x-intercepts} \quad (2+\sqrt{6}, 0) \quad (2-\sqrt{6}, 0)}$$

y-intercept: set $x=0$

$$f(0) = \tfrac{1}{2}(0)^2 - 2(0) - 1$$
$$= -1$$

$$\boxed{\text{y-intercept } (0,-1)}$$

Note: vertex is $\dfrac{-b}{2a} = \dfrac{-(-2)}{2(\frac{1}{2})} = \dfrac{2}{1} = 2 = h$

$$f(2) = \tfrac{1}{2}(2)^2 - 2(2) - 1$$
$$= \tfrac{1}{2} \cdot 4 - 4 - 1$$
$$= 2 - 2 - 1$$
$$= -1 = k$$

(2, -1) vertex
not the x-int or
y-int

NO 12

Together, an experienced word processor and an apprentice word processor can create a document in 6 hours. Alone, the experienced one can do the job 2 hours faster than the apprentice can do it working alone. Find the time for each to do the job alone.

Work rates: fraction done by one alone + fraction done by other alone = fraction done together

together = 6 hrs

experienced = $x - 2$ ← faster = less time

apprentice = x

$$\frac{1}{x} + \frac{1}{x-2} = \frac{1}{6}$$

$$LCD = 6x(x-2)$$

$$\frac{1}{x} \cdot 6x(x-2) + \frac{1}{(x-2)} \cdot 6x(x-2) = \frac{1}{6} \cdot 6x(x-2)$$

$$6(x+2) + 6x = x(x-2)$$

order of op: mult before add

$$6x - 12 + 6x = x^2 - 2x$$

$$12x - 12 = x^2 - 2x$$

$x^2 \Rightarrow$ quadratic set $= 0$.

$$0 = x^2 - 14x + 12$$

12

\- \-

−14

−1, −12

−2, −6

−3, −4

does not factor $\Rightarrow$ quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(12)}}{2(1)}$$

$$x = \frac{14 \pm \sqrt{148}}{2}$$

$$x \approx 13.08, +.91$$

↑

$x-2$ would be negative so this is extraneous

$$x = \frac{14 + \sqrt{148}}{2}$$

$$x = \frac{14}{2} + \frac{2\sqrt{37}}{2}$$

$$x = \boxed{7 + \sqrt{37} \text{ hrs apprentice}}$$

$$x - 2 = 7 + \sqrt{37} - 2$$

$$= \boxed{5 + \sqrt{37} \text{ hrs experienced}}$$

148
2 74
2 37

$$\sqrt{148} = \sqrt{4 \cdot 37} = 2\sqrt{37}$$

If instructions said to "round to nearest tenth of an hour":

$$\boxed{\begin{array}{l} \text{apprentice} = 13.1 \text{ hrs} \\ \text{experienced} = 11.1 \text{ hrs} \end{array}}$$

yes (13)

Beach and Fargo are 400 miles apart. A salesperson travels from Beach to Fargo on Monday, and back to Beach on Tuesday. On Tuesday, she drives 10 mph faster. The total time spent driving was $14\frac{2}{3}$ hours. Find her speed in each direction.

	D	= R	· T
Mon	400	R	$\frac{400}{R}$
Tues	400	R+10	$\frac{400}{R+10}$

$$\frac{D}{R} = \frac{R \cdot T}{R}$$ solve for T

$$\frac{D}{R} = T$$

Total time = time on Monday + Time on Tuesday

$$14\frac{2}{3} = \frac{400}{R} + \frac{400}{R+10}$$

convert to improper $14 \times 3 + 2 = 44$ numerator (same denom)

$$\frac{44}{3} = \frac{400}{R} + \frac{400}{R+10}$$

$$LCD = 3R(R+10)$$

$$\frac{44}{3} \cdot 3R(R+10) = \frac{400}{R} \cdot 3R(R+10) + \frac{400}{R+10} \cdot 3R(R+10)$$

$$\frac{44R(R+10)}{4} = \frac{1200(R+10)}{4} + \frac{1200R}{4}$$

Quick tip: Notice all coefficients are divisible by 4!

$$11R(R+10) = 300(R+10) + 300R$$

$$11R^2 + 110R = 300R + 3000 + 300R$$

$$11R^2 + 110R = 600R + 3000$$

$$11R^2 - 490R - 3000 = 0$$

Notice R^2 quadratic $\Rightarrow$ set $= 0$.

$$R = \frac{-(-490) \pm \sqrt{(-490)^2 - 4(11)(-3000)}}{2(11)}$$

Note:
$b^2 - 4ac$
$(-490)^2 - 4(11)(-3000)$
$372100 = 610^2$ (it does factor)

$$R = \frac{490 \pm \sqrt{372100}}{22}$$

$$= \frac{490+610}{22}, \quad \frac{490-610}{22} = \frac{-60}{11}$$

$$= \boxed{\text{50 mph on Monday}}$$

↑ negative, so this solution is extraneous.

and $R+10 = \boxed{\text{60 mph on Tuesday}}$

Seeing the results, the factors must give:

$x = 50$ and $x = \frac{-60}{11}$

$(x-50)$

$11x = -60$

$(11x+60)$

so we get

$(x-50)(11x+60)$

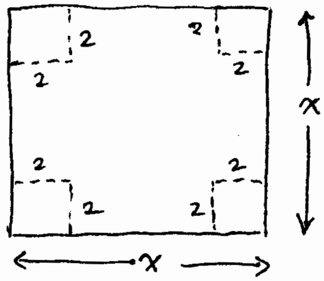
check: $11x^2 + 60x - 550x - 3000$

$11x^2 - 490x - 3000$ ✓

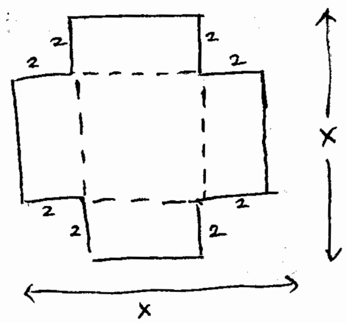
NO 14

Suppose that an open box is to be made from a square sheet of cardboard by cutting 2-inch-by-2-inch squares out of each corner and folding the sides up. The finished box has volume 128 cubic inches. Findt he original dimensions of the sheet of cardboard.

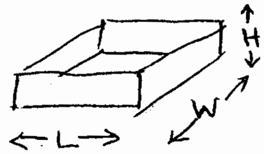
square sheet of cardboard = same dimension for length and width. = x



cut 2-inch squares from each corner
(Do this with a sheet of paper if you're having trouble visualizing it.)



Fold on dotted lines to make a box.
(or a tray).
It is open on top, with no lid.



$V = L \cdot W \cdot H$

Volume of a rectangular solid
Height is 2", the amount cut away.
Length and width are both $x-2-2=x-4$.
(side x, subtract 2 for each missing corner)

$128 = (x-4)(x-4) \cdot 2$

Volume is given 128. Substitute:

$$\begin{cases} V=128 \\ H=2 \\ L=x-4 \\ W=x-4 \end{cases}$$

$64 = (x-4)(x-4)$ Divide both sides by 2.

$64 = x^2 - 8x + 16$ FOIL

$0 = x^2 - 8x - 48$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-48)}}{2}$$ Quadratic formula

$$x = \frac{8 \pm \sqrt{256}}{2}$$

$$x = \frac{8 \pm 16}{2}$$

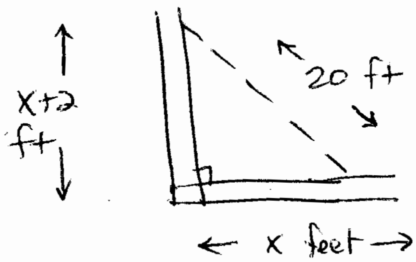
oops! I could have factored!
$(x-12)(x+4)$

$$x = \frac{24}{2}, \frac{-8}{2}$$

$x = 12, -4$ no lengths are negative.

So original cardboard was $\boxed{12 \text{ inches} \times 12 \text{ inches}}$

(15) Students cut across the lawn instead of walking on the sidewalks. How many feet do they save by doing so?



The sidewalks form a right angle, so the path across the grass gives a right triangle.

Pythagorean Theorem

$$a^2+b^2=c^2$$

c must be longest side, or hypoteneuse

$$x^2+(x+2)^2=20^2$$

$$x^2+x^2+4x+4=400 \qquad \text{FOIL}$$

$$2x^2+4x-396=0 \qquad \text{collect on LHS, combine like terms}$$

$$x^2+2x-198=0 \qquad \text{divide by 2}$$

$$x=\frac{-2\pm\sqrt{2^2-4(1)(-198)}}{2} \qquad \text{Quadratic formula}$$

$$x=\frac{-2\pm\sqrt{4+792}}{2}$$

$$x=\frac{-2\pm\sqrt{796}}{2}$$

subtract gives a negative distance, discard this extraneous solution

walk on sidewalks: $x+(x+2)=2x+2$

Substitute: $2\left(\frac{-2+\sqrt{796}}{2}\right)+2$

$$=-2+\sqrt{796}-2$$

$$=\sqrt{796}$$

walk on grass $=20$

Saved $\boxed{\sqrt{796}-20 \text{ ft}}$ (approx 8.2 ft)

YES (16) A water tank can be filled by the large and small inlet pipes in 3 hours. The large inlet pipe can fill the tank in 2 hours less time than the small inlet pipe. Find the time, to the nearest tenth of an hour, that each pipe takes to fill the tank alone.

$$\frac{1}{x-2} + \frac{1}{x} = \frac{1}{3}$$

$$3x + 3(x-2) = x(x-2)$$

$$3x + 3x - 6 = x^2 - 2x$$

$$x^2 - 8x + 6 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 4(6)}}{2}$$

$$x = \frac{8 \pm \sqrt{40}}{2} = \frac{8 \pm 2\sqrt{8}}{2} = 4 \pm 2\sqrt{2} \approx \boxed{6.8\text{ hr}}$$

$\cancel{1.2\text{ hr}}$

NO (17) Bill and his son Billy can clean the house together in 4 hours. Billy takes an hour longer by himself than Bill takes by himself. How long, to the nearest hour, does it take each one to clean the house, if working alone?

$$\frac{1}{4} = \frac{1}{x+1} + \frac{1}{x}$$

$$x(x+1) = 4x + 4(x+1)$$

$$x^2 + x = 4x + 4x + 4$$

$$x^2 - 7x - 4 = 0$$

$$x = \frac{7 \pm \sqrt{7^2 - 4(-4)}}{2}$$

$$= \frac{7 \pm \sqrt{65}}{2} \approx \boxed{7.53\text{ hrs}}$$

$-.5\cancel{3}$ hrs

NO (18) At the 2007 Grand Prix of Long Beach auto race, Simon Pagenaud posted the fastest lap speed, but Sebastian Bourdais won the race. One lap through the streets of Long Beach is 10,391 feet (1.968 miles) long. Pagenaud's fastest lap speed was 0.55 foot per second faster than Bourdais's fastest lap speed. Traveling at these fastest speeds, Bourdais would have taken 0.25 second longer than Pagenaud to complete a lap.

a. Find Sebastian Bourdais's fastest lap speed during the race. Round to two decimal places.
b. Find Simon Pagenaud's fastest lap speed during the race. Round to two decimal places.
c. Convert each speed to miles per hour. Round to one decimal place.

	D	= R	· T
S.P.	10,391	x+.55	$\frac{10391}{x+.55}$
S.B.	10,391	x	$\frac{10391}{x}$
			diff .25

a) $$\frac{10391}{x+.55} = \frac{10391}{x} - .25$$

$$10391x = 10391(x+.55) - .25x(x+.55)$$

$$\cancel{10391x} = \cancel{10391x} + 5715.05 - .25x^2 - .1375x$$

$$\frac{0}{-.25} = \frac{-.25x^2}{-.25} - \frac{.1375x}{-.25} + \frac{5715.05}{-.25}$$

$$0 = x^2 + .55x - 22860.2$$

$$x = \frac{-.55 \pm \sqrt{(.55)^2 - 4(-22860.2)}}{2} = \frac{-.55 \pm \sqrt{91441.1025}}{2}$$

$x = 150.92$ ft/sec

discard (−)

$x + .55 = 151.47$ ft/sec

$$\frac{\text{ft}}{\text{sec}} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \cdot \frac{60\text{ sec}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} = \frac{3600}{5280}$$

c) $150.92\left(\frac{3600}{5280}\right) = \boxed{102.9\text{ mph}}$

$151.47\left(\frac{3600}{5280}\right) = \boxed{103.3\text{ mph Pagenaud}}$

(19) A family drives 500 miles to the Grand Canyon. The return trip was made at a speed 10 mph faster. The total traveling time was $18\frac{1}{3}$ hours. Find the speed to the Grand Canyon and the return speed.

D	= R	T
500	x	$\frac{500}{x}$
500	$x+10$	$\frac{500}{x+10}$

$$\frac{500}{x} + \frac{500}{x+10} = \frac{55}{3}$$

$$LCD = 3x(x+10)$$

$$3 \cdot 500(x+10) + 3 \cdot x \cdot 500 = 55(x)(x+10)$$

$$1500x + 15000 + 1500x = 55x^2 + 550x$$

$$\frac{0}{5} = \frac{55x^2}{5} - \frac{2450x}{5} - \frac{15000}{5}$$

$$0 = 11x^2 - 490x - 3000$$

$$x = \frac{490 \pm \sqrt{(-490)^2 - 4(11)(-3000)}}{2(11)}$$

$$x = \frac{490 \pm \sqrt{372100}}{22}$$

$$x = \frac{490 \pm 610}{22}$$

$$x = 50 \text{ or } -\frac{60}{11}$$

50 mph going to the Grand Canyon
60 mph returning

To derive the quadratic formula, complete the square using $ax^2+bx+c=0$

$$ax^2+bx \qquad = -c$$

collect variables left and constant right

$$a\left(x^2+\frac{b}{a}x \qquad\right) = -c$$

factor out a

$$\# = \frac{1}{2}\cdot\frac{b}{a} = \frac{b}{2a}$$

take half of coefficient of x.

$$\#^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

square result

$$a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right) = -c + a\cdot\frac{b^2}{4a^2}$$

add to LHS inside parentheses

distribute and add to RHS

$$a\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

factor LHS
simplify RHS

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

divide both sides by a.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

write RHS with LCD

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

subtract RHS

$$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

solve by square root property

$$x+\frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$x+\frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}$$

simplify RHS

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

isolate x

Voila! The Quadratic Formula!

$$\boxed{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

add/subtract fractions with LCD.